

BETWEEN “VERY LARGE” AND “INFINITE”: THE ASYMPTOTIC REPRESENTATION THEORY.

A. VERSHIK*

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Abstract

I illustrate the historical roots of the theory which I called later “Asymptotic Representation Theory”, — theory which can be considered as a part functional analysis, representation theory, and more general — probability theory, asymptotic combinatorics, the theory of random matrices, dynamics, etc. The first and very concrete example is a remarkable (and forgotten) paper by J. von Neumann, which I try here to connect with the modern theory of random matrices; the second example is quote of important thought of H. Weyl about the theory of symmetric group. In the last paragraph I give a short review of the ideas of asymptotic representation theory, which was developed starting from 70-s, and now became very popular; I mentioned several important problems, and give (incomplete) list of references. But the reader must remember that this is synopsis of the baby talk.

1 JOHN von NEUMANN: “*Our interest ... is finite but VERY GREAT order of matrices...*”

I start with a beautiful paper by J. von Neumann about the properties of high-order matrices which impressed me very much many years ago.

It happened that the discovery of infinite-dimensional analysis (functional analysis) had followed very promptly the invention of multi-dimensional (but finite) analysis; mathematicians had no time to pay attention to a very important circumstance: the study of very high

*St. Petersburg Department of Steklov Institute of Mathematics. E-mail: avershik@gmail.com. Supported by the grants RFBR 11-01-12092-ofi-m and RFBR 11-01-00677-a.

dimensions must precede that of infinite-dimensional objects; the understanding of “intermediate” (between finite and infinite) things is absolutely necessary for the understanding of the actual infinity. The exciting perspectives and fascinating results of the functional analysis in the first part of the 20th century eclipsed the beauty of high-dimensional analysis. My viewpoint on this was described in [19]. Many mathematicians, including the great creators of functional analysis forewarn of that. I want to mention at least two of such warnings, which impressed me very much when I was a young mathematician and had just studied functional analysis.

I will tell about remarkable John von Neumann’s paper “Approximative properties of matrices of high finite order” ([1]).¹ In the volume of Russian translation of the Selected papers by von Neumann in functional analysis (see [2]) I gave some comments about this article and now I will use part of those remarks with some new points. This is a remarkable paper. I think, it is almost forgotten and is cited relatively rare in the mathematical literature in comparison with the other papers of this author, but it deserves much more attention. I remember only Glimm’s reference to it in his famous paper [3] on types of factors. The main message of the paper is in the following claim:

“Our interest will be concentrated in this note on the conditions in H_n and M_n (Hilbert space and unit ball in matrix norm of dimension n — comments by AV) when is finite but VERY GREAT. This is an approach to the study of infinite dimensional which differ essentially from the usual one. The usual approach consists in studying an actual infinite dimensional unitary space i.e. Hilbert space H_∞ as done. We wish to investigate instead the ASYMPTOTIC behavior of H_n and M_n for finite n when $n \rightarrow \infty$. We think that the latter approach has been unjustifiably neglected as compared with the former one. It is certainly not contained in it, since it permits the use of the notion of norm (Hilbert-Schmidt norm — AV) and normalized trace which owing to factor $1/n$ possesses no analogs in H_∞ . Since Hilbert space H_∞ was considered as a limiting case of H_n for $n \rightarrow \infty$, we feel that such a study is necessary in order to clarify to what extent H_∞ is or is not the only possible limiting case. Indeed we think that it is not, and that investigation on operator rings by F.J.Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.”

Here we hear the enthusiasm of the author with his striking idea of the continuous dimension, factors of type II_1 etc., which were revolutionary at that time. But even outside of those ideas the point of view which proclaimed by von Neumann is extremely important. The main result of this paper is very important even now. The author established a very interesting property of the matrices $M_n(\mathbb{C})$ of big order which was unknown before and even now little known. The main result illustrates how incomplete is our knowledge of the multi-dimensional theory of matrices; in a sense, it is much more modest than our knowledge of the infinite-dimensional case. I quote the main statement of paper here.

¹One of the reason why this article was not so long known, perhaps is the strange place of publication - Portugal Mathematical Journal, - and difficult time — time of the second World War- 1942. Remark that von Neumann before had published one of his important paper in the more strange Journal — “Proceeding of Tomsk University” (1937). Very important paper on another classic of functional analysis Leonid Kantorovich “On transportation mass” was also had published in 1942 and so was not known in the West for a long time (see [?])

Theorem 1. (theorem 9.6 in [1]). For every $\delta \in (0, \frac{1}{2})$ there exists $\epsilon = \epsilon(\delta) > 0$ such that if P_E is an orthogonal projection on the subspace E with property:

$$\delta < \frac{\dim E}{n} < 1 - \delta,$$

then for every positive integer n there exists a matrix $A \in M_n(\mathbb{C})$ with $\|A\| \leq 1$ such that

$$\|AP_E - P_EA\| > \epsilon.$$

This means that for all n in the space \mathbb{C}^n there exists a operator A for which there are no subspaces of the intermediate dimensions which are almost invariant together with its orthogonal complements. This is a fact from the geometry and linear algebra of the *Hilbert space*. This result has no literal interpretation in the infinite-dimensional case, because the step from finite to infinite dimension changes drastically the notion of genericity. In the paper as usual for this author, there are many various reformulations of the main result. I suggest below some strengthened formulation and discussion using the point of view of the modern theory of random matrices.

Remarks. 1. The method of the proof was in fact based on the more or less direct calculations of the Lebesgue measure of the set of matrices which do not have the property formulated in the theorem and have Hilbert-Schmidt norm less or equal than one. Is clear from the calculations that this measure is much less than the measure of all unit ball in $M_n(\mathbb{C})$. Von Neumann called the method as “volumetric method,” we can call it “the entropy method”. Most difficult part of the paper devoted to the proof of the fact that ball in Hilbert-Schmidt norm can be substitute on the ball in spectral norm; the direct calculation of the asymptotic of the measure unit ball in the spectral norm was failed to calculate.

2. It is interesting that the problem appeared when von Neumann tried to construct an example of a new type of hyper-finite II_1 factors using the result of the paper. But soon (as it mentioned in the proofs of the given paper) he (with F.Murray) had proved that there are exist non-isomorphic factors of type II_1 using another method. Moreover, one cannot obtain new examples of the factors on the way of this paper, because the hyper-finite factor of type II_1 is unique up to isomorphism, this was proved later, meanwhile the method above can give only hyper-finite ones. Nevertheless the result of the paper has independent interests. As the author emphasized, the set of required matrices in the theorem was not constructively defined, and he could not give a constructive proof of the existence of a such matrix. I hope that after more than 70 years it is now possible to refine and to give more constructive and a simpler proof of von Neumann’s result. I will mention only some interpretation of the result.

Of course, these matrices cannot be unitary or even normal. Recall that almost all, with respect to the Haar measure, elements of $\text{GL}(n, \mathbb{C})$ are semi-simple and have distinct eigenvalues and, as it was used in the paper, can be represented as upper-triangle (semi-diagonal as von Neumann called it) matrices with different elements on the diagonal. With respect to the (infinite) Haar measure m , we can say that the almost all matrices are semi-simple and has this form. From other side we can speak equivalently about the random

choice of the linear independent n -vectors in n -dimensional vector spaces or point on non-orthogonal Stiefel manifold. So we can consider the above von Neumann's result from the point of view of the theory of random matrices or random repairs. Why has the theory of random matrices appeared here? Because we have dealt with a theorem about the statistics and properties of the geometry of the configuration of the eigenvectors of an arbitrary finite-dimensional semi-simple random operator. Many papers are devoted to the study of the spectrum of random matrices in various situations, but I do not know of the investigations of the asymptotic of interesting functionals on non-unitary and non-hermitian matrices. This statistics of the eigenvectors for general matrices is very intriguing subject. These questions are much deeper than the circle law for the spectra of non-self-adjoint matrices.

I want slightly strengthen the result of the paper in the following conjecture.

Conjecture 1. *Let $\delta > 0$, and let $\text{Gras}_\delta = \bigcup_{k:\delta < \frac{k}{n} < 1-\delta} G_{n,k}$, where $G_{n,k}$ is the Grassmannian of all k -dimensional subspaces in \mathbb{C}^n . Then there exists $\epsilon = \epsilon(\delta) > 0$ such that for all n ,*

$$\int_{A \in \text{GL}_1} \min_{E \in \text{Gras}_\delta} \|[A, P_E]\| dm(A) = \epsilon > 0,$$

where $\text{GL}_1 = \{A \in \text{GL}(n, \mathbb{C}) : \|A\| \leq 1\}$; P_E is the orthogonal projection to the subspace E , and m is the normalized Haar (Lebesgue) measure on the ball $\text{GL}(n, \mathbb{C})_1$, $\|\cdot\|$ is the normalized Hilbert-Schmidt norm in $M_n(\mathbb{C})$, and $[A, P] = AP - PA$.

It is additional interesting question, if there exists a limit distribution of the integrand as a function on $\text{GL}(n, \mathbb{C})_1$; what is it and what is the infinite-dimensional interpretation of it?

Additional task is to prove this conjecture for spectral norm: it seems that the method of the paper perhaps allow to do this. Instead of Lebesgue measure it is natural to use Gaussian measure on the space of matrices. Moreover, we can use only Borel (upper diagonal) subgroup of $\text{GL}(n, \mathbb{C})$ instead of the space of all matrices. Another thing to use instead of a matrix A , a configuration (repair) of n vectors eigenvectors of matrix A , and integrate over non-orthogonal Stiefel manifold which is the space of all nondegenerated repairs. We will not stay on this here.

The conjecture strengthens von Neumann's result in the following sense: instead of the supremum of norms we considered the integral of commutators over all matrices. In more expressive form we can rewrite Conjecture equivalently:

$$\inf_{n \in \mathbb{N}} \int_{A \in \text{GL}_1(n, \mathbb{C})} \min_{E \in \text{Gras}_\delta} \{\|P_E A P_{E^\perp}\| + \|P_{E^\perp} A P_E\|\} dm(A) \equiv c = c(\delta) > 0,$$

It is interesting to find the limit distribution of integrand (when n tends ∞); it can happened that limit measure is a delta measure.

There are similar problems in the spirit of asymptotic behavior of the random matrices which one can formulate as extension of von Neumann questions from that remarkable paper. We will return to this.

2 HERMANN WEYL: “THE GOAL OF COMBINATORICS IS TO FIND PROPERTIES OF TYPICAL PERMUTATIONS OF LARGE FINITE DEGREES”

Now I will mention H. Weyl’s opinion on the role of asymptotic, concretely, about symmetric groups of high order. In a sense, this idea is also in the spirit of those by von Neumann and to asymptotic representation theory. After that I will describe the main problems that I consider as a kind of agenda for the future development of this theory. The next example is similar to the previous one, but has a different flavor. In his book *Philosophy of Mathematics and Natural Sciences*, H. Weyl (see [4]) wrote a special chapter about combinatorics and made a perspicacious remark about it: “*Perhaps the simplest combinatorial entity is the group of permutations of n objects. This group has a different constitution for each individual number n . The question is whether there are nevertheless some asymptotic uniformities prevailing for large n or for some distinctive class of large n .*” He then continued: “*Mathematics has still little to tell about such a problem.*”

I used this quote as an epigraph to my talk [18] at the International Congress of Mathematicians in Zürich in 1994 and, earlier, in the paper [26]. One can still consider this as a program for the future. But, nevertheless, now we can say more than in Weyl’s time. In the last quarter of the 20th century, the status of combinatorics changed drastically, because its new aspects appeared under the influence of statistical physics, representation theory, geometry.

I want to illustrate this with examples from the theory of representations of symmetric groups.

One of the main objects in combinatorics are Young diagrams (or Ferrers diagrams). The link between Young diagrams and theory of symmetric groups is attributed to Alfred Young. In papers by F. Frobenius, I. Schur, A. Young, the main facts were established, including formulas for the characters, the branching rule, a link to $GL(n, R)$, etc. The remarks and improvements by J. von Neumann, H. Weyl, L. Brauer created the representation theory of symmetric groups, which existed in this form for almost 100 years. But some important questions were neglected and stayed open. H. Weyl’s observation shows only one of these problems.

I was never satisfied with the standard explanation why Young diagrams appear in this theory, which is that the branching rules are the same for the irreducible representations of the symmetric groups and for the Young graph. But the true explanation came from the inductive, or asymptotic, point of view.

We must pay attention to Coxeter’s description of symmetric groups and to the role of the so-called Gelfand–Tsetlin commutative subalgebra of the group algebra of a symmetric group. The latter algebra (which is generated by the centers of the group algebras $\mathbb{C}[S_k]$, $k = 1, \dots, n$) appeared, in a sense, from viewing S_n as the union of the inductive family $S_1 \subset S_2 \subset \dots \subset S_n$. Its classical analog is the Gelfand–Tsetlin basis for representations of the groups $U(n), O(n)$. But it is just the “asymptotic” point of view that I tried to use for the symmetric groups.

In a finite form, it was described in my joint paper with A. Okounkov [12], which contained the realization of the previous idea about new views on the representation theory of finite symmetric group [11]; see also [13, 14] and later development of those ideas which had been used in many papers as a background for the representation theory of the symmetric groups, f.e.[15]. I consider this story as a good example of how fruitful is influence of the “almost infinite” philosophy on finite problems.

On the other hand, in the study of the representation theory of the infinite symmetric group S_∞ , some aspects of the asymptotic theory of finite groups turn out to be very useful, although these subjects are completely different.

3 HOW TO CHOOSE RIGHT INFINITE OBJECTS THAT GENERALIZE A FINITE ONE.

The idea about Asymptotic Theory of Representations (it is possible to use abbreviation ART) appeared in my mind in the beginning of 1970-s. I thought about infinite dimensional groups and algebras in comparison with ergodic theory, and combinatorics. The first contribution was the attack to the problem of limit behavior of joint distribution of the cycles of a random permutation. The results was published in the short announcement [25], communicated by Yu.Linnik (1972) who was extremely interested with the connection of this type of asymptotic and number theory, and later two big articles [26] which was written in 1974. Then I had more or less precise plan which included the study of the representations of symmetric groups and analysis of the paper E.Thoma ([16]) about the characters of infinite symmetric group which was recommended to me by I. M. Gel’fand. My student (aspirant) of that time Sergey Kerov prepared his thesis (candidate dissertation) on the theme about variant of duality theory of the algebras with involution which I suggested in 1972. When he had finished his thesis I suggested him to work with me on that which I called later as ART. It was a successful choice and we started to work together on this topic in the middle of 1970s. The main idea of my approach developed jointly with S. Kerov was to obtain the properties of the group S_∞ and its representations as asymptotic properties. One of the best examples here was the proof of Thoma’s formula for characters obtaining them as the limits of sequences of irreducible characters of the finite groups S_n as $n \rightarrow \infty$. In principle, this can be done by the ergodic method (ergodic theorem), but the calculation is rather involved (see [20, 5, 7, 8, 9]). A more difficult question concerns the study of the corresponding factor representations, their realizations and analytical properties. There are remarkable papers by S Kerov, G. Olshansky, A. Okounkov, A. Borodin on the topic, and now we have much information about these problems, but some of the important questions (for instance, about representations of type II_∞) are still open (see [24]).

The several central problems in the theory of locally semi-simple algebras and inductive limits of groups are as follows.

1. To find the list of traces e.g. central measures on the space of paths of the Bratteli diagram, or — invariant measures for the so called adic transformation defined by the lexicographic ordering of paths of the Bratteli–Vershik.

2. To describe the K_0 -functor as a Riesz group = the ordered structure on the Grothendieck group.

3. To give various realization of the main classes of the representations of the corresponding group and algebras, and to give asymptotic interpretation of the corresponding results of the classical theory. We have very few examples of results of that type. One of that in [17]. We hope that understanding of some classical procedures like Bethe ansatz will come from asymptotic representation theory.

The first two problems were solved only for few algebras and groups. For the group S_∞ , we have several proofs of Thoma's theorem on the list of characters (Thoma's [16], Vershik and Kerov's [20], Olshanski and Okounkov's [10]), but still have no pure combinatorially-probabilistic proof which should be applied not only to Young graph, but also to a wide class of AF -algebras, including Hasse diagrams of the distributive lattices and so on, (see recent paper [27] which contains a new ideas about invariant measures).

The structure of the K_0 -functor for S_∞ was discovered in [21], but still has no serious applications. Both problems belong to asymptotic representation theory, as well as to combinatorics and probability theory.

The asymptotic study of S_∞ and other similar groups G (like the infinite unitary group $U(\infty)$ and other inductive limits of sequences of finite or compact groups) has been concentrated in the last years near the so-called harmonic analysis, which was formulated by G. Olshanski as a bunch of problems about the decomposition of a natural representation of the double group $G \times G$ into a spectrum of irreducible representations (the so-called theory of z -measures). Studying representations of the double (left and right) group as a method of studying representations of the group G itself was suggested by von Neumann. A crucial fact is that the restriction of a representation of the double group to the left or right component is a representation of type II; in particular, in the case of irreducible representations of the double group, it is a factor of type II_1 or II_∞ . In order to single out representations of the double group that are useful in this sense, in the case of locally finite groups G . Olshanski introduced the notion of admissible representations. This is a special class of representations of $G \times G$, and for some groups G (or locally semisimple algebras) that are not of type I, the category of representations of $G \times G$ is of type I. For the infinite symmetric group, the description of this category was started by G. Olshanski and continued by A. Okounkov [10]. Nevertheless, the analysis is not yet completed; see the analysis of the types of factors in [24]. The corresponding analysis for $U(\infty)$ and other groups is only at the first stage.

The picture is different in the case of the group of infinite matrices over finite fields. It started with the paper [22] and then was continued in [23, 6]. This theory is related to the classical results on representations of the group $GL(n, F_q)$ (R. Green, D. Faddeev, A. Zelevinsky). I want to emphasize here only that the similarity between representations of this group and representations of the infinite symmetric group extends only to the so-called principal series of representations, and even in this case we have only first results.

The last remark concerns another asymptotic question in the same spirit: how to correctly generalize the classical finite concept of the Schur–Weyl duality between representations of the symmetric group S_N and the group $GL(n, C)$ to the infinite case? As far

as I know, in our paper [17] this question is discussed for the first time. Among many possibilities, the authors have chosen one; it will be clear later on whether this choice is justified.

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