

POLYMORPHISMS, MARKOV PROCESSES, AND QUASI-SIMILARITY

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To the centenary of my mother H. Ya. Lusternik

ABSTRACT. In this paper we develop the theory of polymorphisms of measure spaces, which is a generalization of the theory of measure-preserving transformations. We describe the main notions and discuss relations to the theory of Markov processes, operator theory, ergodic theory, etc. We formulate the important notion of quasi-similarity and consider quasi-similarity between polymorphisms and automorphisms.

The question is as follows: is it possible to have a quasi-similarity between a measure-preserving automorphism T and a polymorphism Π (that is not an automorphism)? In less definite terms: what kind of equivalence can exist between deterministic and random (Markov) dynamical systems? We give the answer: every nonmixing prime polymorphism is quasi-similar to an automorphism with positive entropy, and every K -automorphism T is quasi-similar to a polymorphism Π that is a special random perturbation of the automorphism T .

1. Introduction.

1.1. The theory of polymorphisms as a generalization of ergodic theory.

The simplest example of a polymorphism with invariant measure that is not a measure-preserving, auto- or endomorphism is the “map” from the unit circle S^1 with Lebesgue measure to itself whose graph is the following cycle (curve) on the 2-torus $S^1 \times S^1$ (also with Lebesgue measure):

$$\{(u, v) : u^2 = v^3, |u| = |v| = 1\}.$$

Another example is a “random automorphism,” or the convex combination $p_1T_1 + \dots + p_nT_n$, where T_i , $i = 1, \dots, n$, are automorphisms of a measure space, $p_i > 0$, $\sum p_i = 1$.

In these examples, each point of the space under consideration has many images (a random image) and a random preimage. A general polymorphism with invariant measure from one measure space to another is, by definition, a *measure in the product of these spaces with given marginal projections* (see Definition 1). It was first defined in the paper [16]. The dynamics of such “maps” is an interesting open area.

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In the first part of the paper we give a brief survey of definitions and important properties of the notion of polymorphism. Briefly speaking, we *generalize the foundations of ergodic theory to a natural measure-theoretic version of the dynamics of multivalued maps*. We called the main objects of our theory *polymorphisms*; a polymorphism can be regarded simply as a *Markov map with a fixed invariant measure*, or the two-dimensional distribution of a stationary Markov process, or a bistochastic measure, or a joining of measure spaces.

Parallel notions to the notion of polymorphism in other parts of mathematics are: *correspondence* in algebra and algebraic geometry; *bifibration* in differential geometry, *Markov map* in probability theory, *Young measure* in optimal control, etc. The notion of polymorphism (also with a quasi-invariant measure) generalizes all such examples, see Section 2.3 (“Why polymorphisms?”), but we are mainly interested in the geometry and dynamics of polymorphisms in the framework of *measure theory*.

From the point of view of dynamics and physics, the notion of polymorphism corresponds to the “*coarse-graining*” approach to dynamics: instead of one-to-one maps we are allowed to consider maps that send a point to a measure, or elements of a partition to a family of elements of the same partition (a set of “grains” to itself, see Section 2). This opens new possibilities, which are forbidden in the classical theory: for example, we can define the notion of the quotient of an automorphism by a partition *that is not invariant under the automorphism*; this quotient is a polymorphism, but not an automorphism. Another important direction is approximation of automorphisms with polymorphisms, especially with finite polymorphisms; this approach is an alternative to Rokhlin approximations and is perhaps more effective for automorphisms with positive entropy.

The set of polymorphisms of a given measure space has a rich structure: it is a convex weakly compact topological semigroup whose invertible elements are measure-preserving automorphisms. The functional analog or operator formalism of our theory is the theory of Markov operators in the Hilbert space $L^2_\mu(X)$. A *Markov operator is a positive contraction that preserves constants*; the positivity means preserving the cone of nonnegative functions in the space $L^2_\mu(X)$. This immediately leads us to the necessity of generalizing the theory of contractions and non-self-adjoint operators in Hilbert spaces to Markov operators.

The spectral theory of concrete Markov operators had been studied for a long time; but the theory of contractions in Hilbert spaces apparently did not up to now attract the attention of specialists in dynamical systems; it gives a further development of spectral theory and a new type of questions in dynamics, which is essentially important for polymorphisms. We choose one of such questions that is interesting in itself and describe it briefly in the next section.

1.2. The problem of quasi-similarity of automorphisms and polymorphisms, and paradoxical Markov processes. Recall (see [10]) that a bounded operator T in a Hilbert space H_1 is called a *quasi-image* (“quasi-affinitet” in [10]) of a bounded operator S in a Hilbert space H_2 if there exists a continuous linear operator L from H_1 to H_2 that may have no bounded inverse but has a **dense image** (so that the inverse operator is defined on a dense set) such that $LT = SL$. Two bounded operators are called *quasi-similar* if each of them is a quasi-image of the other. Quasi-similarity is an equivalence in the space of bounded operators, which is weaker than similarity (or unitary equivalence) of operators; it may happen that there is an equivalence between a unitary operator and a proper contraction

operator that is totally nonunitary (see [10]). This equivalence does not in general preserve spectra of operators. Such examples are important in functional analysis and scattering theory.

Quasi-similarity for Markov operators and the parallel geometric notion of quasi-similarity for measure-preserving transformations and especially for polymorphisms seems to have never been considered systematically. A polymorphism Π_1 is a quasi-image of a polymorphism Π_2 (in particular, one or both of them can be automorphisms) if there exists a dense (in the sense of Section 2) polymorphism Λ such that $\Lambda \cdot \Pi_1 = \Pi_2 \cdot \Lambda$. Equivalently, this means that there exists a joining between Π_1 and Π_2 . *Two polymorphisms are quasi-similar if each of them is a quasi-image of the other.* The question is as follows: can such a Λ exist if Π_1 is an automorphism and Π_2 is a proper polymorphism? The general problem is **to describe all quasi-similar pairs “automorphism \leftrightarrow polymorphism.”**

In other words, the problem is to describe pairs “deterministic transformation \leftrightarrow random transformation” that can be quasi-similar in the above sense. It may happen that this question can be related to the long discussion among physicists (see, e.g., [9, 3, 4]) on possible equivalence between deterministic and random systems.

In a nontrivial case, a polymorphism must be “prime,” i.e., have no factor endomorphisms, and also nonmixing in the sense of the theory of Markov processes; otherwise the problem is not interesting. Even the existence of such polymorphisms is not obvious. The first example was given in probabilistic terms by M. Rosenblatt (see [14, Ch. 4.4]). Then a smooth example was suggested in [17]; it was a random perturbation of a hyperbolic automorphism of the torus. In this paper we give a formulation of the complete solution of the problem.

Here we present one result in this direction and leave the general case for another publication (see Section 5).

1. *There is an automorphism T with positive entropy that is a canonical quasi-image of a nonmixing prime polymorphism Π ; if the conjugate polymorphism Π^* is also prime and nonmixing, then the polymorphism Π is quasi-similar to T .*

2. *Assume that for a K -automorphism T there exists a finite or countable K -generator such that in the symbolic realization of T with this generator, the homoclinic equivalence relation is ergodic; then there exists a polymorphism Π that is quasi-similar to T , more exactly, the following weak limits exist and define two intertwining polymorphisms*

$$\Lambda_1 = \lim_{n \rightarrow \infty} \Pi^n T^{-n}, \quad \Lambda_2 = \lim_{n \rightarrow \infty} T^n (\Pi^*)^n$$

that realize the quasi-similarity:

$$\Pi \cdot \Lambda_1 = \Lambda_1 \cdot T, \quad \Lambda_2 \cdot \Pi = \Pi \cdot \Lambda_2.$$

The polymorphism Π is a special random perturbation of the K -automorphism T .

The question of whether this K -automorphism T is unique leads to a very interesting problem for automorphisms and especially K -automorphisms: *do there exist two automorphisms that are not isomorphic but are quasi-similar?* I do not know the answer.

In order to explain our method, we must say several words about stationary Markov processes with paradoxical property that appeared in this problem.

It is well known that a mixing Markov chain is regular in the sense of Kolmogorov (or pure nondeterministic): the σ -field of the infinite past is trivial. On the other hand, mixing is equivalent to irreducibility, i.e., the absence of nontrivial partitions

of the state space into subclasses, or the absence of deterministic factors of a Markov process (or primality for polymorphisms).¹

Contrary to this, the absence of nontrivial (measurable) partitions into subclasses, or the absence of nontrivial deterministic factor processes, which we called primality for polymorphisms, does not imply mixing and regularity for general Markov processes — there exist nonmixing Markov processes that have no deterministic factors; we called such paradoxical Markov processes *quasi-deterministic* Markov processes.

The reason of this difference between processes with discrete and continuous state spaces is rather deep and relates to the theory of measure-theoretic equivalence relations. A *measurable* partition of the state space of a Markov chain into “subclasses” allows us to decompose the Markov chain into irreducible mixing (nonhomogeneous in time) chains. But in the general case it may happen that there exists a **non-measurable** partition of the state space, or an ergodic equivalence relation that is invariant under the polymorphism, so that there is no regularity, but at the same time there are no deterministic factors. This effect underlines quasi-similarity. As we will see, such processes have a hidden determinism: you cannot predict the value of the process at time zero with probability one if you know the infinite past, but you can define a conditional probability on this state space, and this conditional probability is different for different points of the tail space (“entrance boundary”). The action of the shift on the tail σ -field gives an action on the set of these conditional measures.

The role of the K -automorphism mentioned in the theorem is played by the “tail shift” — the restriction of the shift onto the tail σ -field; the Markov generator is not a K -generator for the Markov shift; nevertheless, this is a K -shift. In order to prove the K -property, we must change the generator. All details will be published in a separate paper.

In the second section we give the first definitions, examples, links, etc. In the third section we introduce the operator formalism and operator version of quasi-similarity. Section 4 is devoted to the corresponding Markov processes, tail σ -fields, residual automorphisms. The main results are formulated in Section 5.

Note that we consider the case when the time is \mathbb{Z} , but there are no serious obstacles to extending the results to the continuous time \mathbb{R} . We will return to this topic in more detail elsewhere.

2. Definitions and properties of polymorphisms. We will briefly define the main notions we need. Some details can be found in [16].

The notion of polymorphism is a measure-theoretic analog of what people called a multivalued map. In the framework of measure theory, the value of a “multivalued map” at a point is not a subset of the target space, but a measure on this space. In this sense, a polymorphism is a measure-theoretic analog of a Markov map; in the subsequent sections we will discuss the relation to the theory of Markov processes in detail.

Objects similar to polymorphisms have many names in various theories (see the introduction). Our considerations are directed towards *dynamics, probability, and ergodic theory*.

¹Perhaps, this fact was first proved in the paper [20] by a student of Kolmogorov.

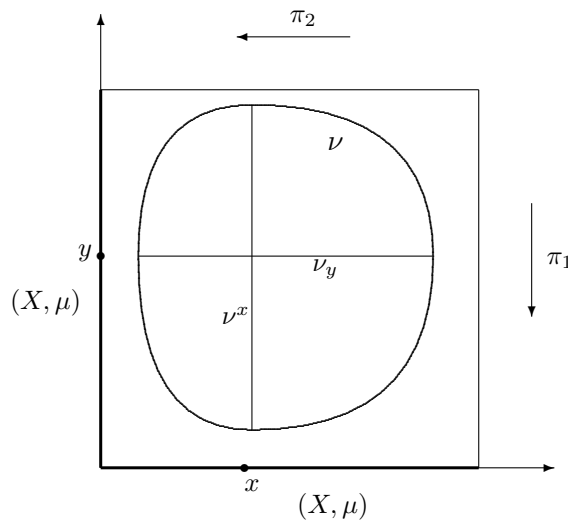
2.1. **First definitions.** Let (X, μ) be a Lebesgue space with continuous measure μ (i.e., a measure space isomorphic to the unit interval with the Lebesgue measure).

Definition 1. A *polymorphism* Π of the Lebesgue space (X, μ) to itself with invariant measure μ is a diagram consisting of an ordered triple of Lebesgue spaces:

$$(X, \mu) \xleftarrow{\pi_1} (X \times X, \nu) \xrightarrow{\pi_2} (X, \mu),$$

where π_1 and π_2 stand for the projections to the first and second component of the product space $(X \times X, \nu)$, and the measure ν , which is defined on the σ -field generated by the product of the σ -fields of mod0 classes of measurable sets in X , is such that $\pi_i \nu = \mu, i = 1, 2$.

The measure ν is called the *bistochastic measure of the polymorphism* Π .



A polymorphism Π^* is called *conjugate* to the polymorphism Π if its diagram is obtained from the diagram of Π by reflecting with respect to the central term.

Consider the “vertical” partition ξ_1 and the “horizontal” partition ξ_2 of the space $(X \times X, \nu)$ into the preimages of points under the projections π_1 and π_2 , respectively. In terms of bistochastic measures, the *value of a polymorphism at a point* $x \in X$ is a conditional measure. More precisely, we have the following definition.

Definition 2. In the above notation, the *value* of the polymorphism $\Pi : X \rightarrow X$ at a point $x_1 \in X$ is, by definition, the conditional measure ν^{x_1} of ν on the set $\{(x_1, \cdot)\}$ with respect to the vertical partition ξ_1 (the transition probability); similarly, the *value* of the conjugate polymorphism Π^* at a point $x_2 \in X$ is the conditional measure ν_{x_2} of ν on the set $\{(\cdot, x_2)\}$ with respect to the horizontal partition ξ_2 (the cotransition probability). These conditional measures are well-defined on sets of full measure.

Remark. It is very important that the set of conditional measures $\{\nu^x, x \in X\}$ itself does not determine the polymorphism — we need to know also the measure μ on X . Recall that an ordinary Markov map is determined by the list of transition probabilities.

With obvious modifications, we can define more general notions:

1) a polymorphism of one measure space (X_1, μ_1) to another measure space (X_2, μ_2) :

$$(X_1, \mu_1) \longleftarrow (X_1 \times X_2, \nu) \longrightarrow (X_2, \mu_2),$$

where the measure ν have marginal projections μ_1 and μ_2 ;

2) a polymorphism with *quasi-invariant measure*; in this case, the projections $\pi_1\nu$ and $\pi_2\nu$ of the measure ν are equivalent (not necessarily equal) to the measures μ_1 and μ_2 , respectively.

For the most part, we will consider polymorphisms of a space with continuous measure to itself with a finite invariant measure.

All notions should be understood up to set of zero measure (mod 0). In fact, our objects and morphisms are classes of coinciding mod 0 objects and morphisms, but we will not repeat the corresponding routine comments when this does not cause any problem.

For simplicity, we assume that the σ -field A of the Lebesgue space (X, μ) has a countable basis B . This means that on X we have the standard Borel structure; consequently, on the space $X \times X$ we have the countable basis $B \times B$, the standard Borel structure, and the σ -field generated by this basis. Thus all bistochastic measures ν corresponding to polymorphisms of the space (X, μ) to itself will be defined on this σ -field.² Now the set of all bistochastic measures becomes an affine compact space equipped with the topology of weak convergence on the basis.

2.2. Further definitions and properties. The following proposition-definition describes structures on polymorphisms.

Proposition 1. *The set of polymorphisms (bistochastic measures) \mathcal{P} is a topological semigroup with the following natural product. Let Π_1, Π_2 be two polymorphisms with bistochastic measures ν_1, ν_2 ; then the product $\Pi_1\Pi_2$ has bistochastic measure ν defined by*

$$\nu^x(A) = \int \nu_1^y(A) d\nu_2^x(y).$$

The semigroup \mathcal{P} has the zero element: this is the polymorphism Θ with bistochastic measure $\nu = \mu \times \mu$; obviously, $\Theta \cdot \Pi = \Pi \cdot \Theta = \Theta$ for every Π ; we call Θ the zero polymorphism.

The set of polymorphisms \mathcal{P} has the structure of a semigroup with involution $\Pi \rightarrow \Pi^$, which was defined above.*

The subgroup of invertible elements of the semigroup \mathcal{P} is the group of measure-preserving transformations; the semigroup of measure-preserving endomorphisms is a subsemigroup of \mathcal{P} ; the bistochastic measure corresponding to an endomorphism T is the measure on the set $\{(x, Tx)\}_{x \in X} \subset (X \times X)$ that is the natural image of the measure μ under the map $x \rightarrow (x, Tx)$.

All assertions of the proposition are obvious. We explain only the last one. Assume that T is an endomorphism of a Lebesgue space (X, μ) with invariant measure. Consider the graph of T , i.e., the set $\{(x, Tx)\}_{x \in X} \subset X \times X$, and the measure μ on this graph (more rigorously, we identify a point $x \in X$ with the point (x, Tx) , so that the measure ν can be regarded as the image of the measure μ on the graph of T). Since T is a measure-preserving map, it follows that ν is a bistochastic measure, and we can identify the endomorphism T with the corresponding polymorphism.

²We omit the discussion of the nontrivial question concerning the independence of such a definition on the choice of the basis B .

Thus we embed the semigroup of endomorphisms and, in particular, the group of automorphisms into the semigroup of polymorphisms.

Let us define the notions of factor polymorphism, ergodicity, mixing, etc., and compare them with the same notions for endomorphisms.

1. A measurable partition ξ is called *invariant* under a polymorphism Π if for almost all elements $C \in \xi$ there exists another element $D \in \xi$ such that for almost all (with respect to the conditional measure on C) points $x \in C$, we have $\mu^x(D) = 1$, where μ^x is the Π -image of x . In other words, the factor polymorphism Π_ξ of Π by an invariant partition ξ is an endomorphism of the space (X_ξ, μ_ξ) .

In particular, if for almost all elements $C \in \xi$ of ξ we have $\mu^x(C) = 1$ for almost all $x \in C$, then the partition ξ is called a *fixed* partition for Π and the corresponding factor polymorphism is the identity map on X_ξ .

Both definitions restricted to endomorphisms give the corresponding notions (of invariant and fixed partitions) of ergodic theory. Any polymorphism has the maximal fixed partition and can be decomposed into the direct integral of ergodic components over this partition.

2. The *ergodicity* of a polymorphism Π means the absence of fixed measurable partitions, in other words, the absence of identical factors.

3. Given a polymorphism Π of a space (X, μ) with bistochastic measure ν , the *factor polymorphism* of Π by a measurable partition ξ is the polymorphism of the space $(X/\xi, \mu/\xi)$ to itself with bistochastic measure $\nu/(\xi \times \xi)$. That is, we have the diagram

$$(X_\xi, \mu_\xi) \longleftarrow (X_\xi \times X_\xi, \nu_{\xi \times \xi}) \longrightarrow (X_\xi, \mu_\xi).$$

Thus the factor polymorphism of any polymorphism by any measurable partition does exist, in particular, the *factor polymorphism of any automorphism by any (not necessarily invariant) partition always exists*.

4. A polymorphism is called *prime*³ if it has no nontrivial invariant partitions.⁴ Prime nonmixing polymorphisms are the main objects of the second part of this paper. In a sense, primality is an analog of Rokhlin's notion of exactness for endomorphisms.

5. A polymorphism Π is called *mixing* if the sequence of its powers tends to the zero polymorphism (see above) in the weak topology: $w\text{-}\lim_{n \rightarrow \infty} \Pi^n = \Theta$. Note that it may happen that a polymorphism is mixing while its conjugate is not.

This notion of mixing has nothing to do with the notion of mixing in ergodic theory: for example, no automorphism is mixing in our sense. We use this term, because it is equivalent to the traditional notion of mixing in the sense of the theory of Markov processes — see the next section.

6. We say that a polymorphism Π is *injective* if the partition into the preimages of points under the map $x \rightarrow \Pi(x)$ coincides with ε (= partition into separate points mod 0). We will deal with injective polymorphisms in Section 5. If the measures $\Pi(x)$ are discrete (finitely supported) for almost all points x , we say that the polymorphism Π is of *discrete rank* (respectively, of *finite rank*). If the

³In [17], this notion was called "exactness."

⁴There is a small difference between this notion and the notion of a prime automorphism, which is, by definition, an automorphism that has no invariant partitions except the trivial partition ν and the partition into separate points ε ; thus a prime automorphism is not a prime polymorphism in our sense, because ε is an invariant partition for it.

bistochastic measure of a polymorphism Π of a space (X, μ) is absolutely continuous with respect to the product measure $\mu \times \mu$, we say that Π is *absolutely continuous*.

7. A polymorphism Π called *dense* if there is no nonzero measurable function f that has zero mean with respect to μ -almost all (in x) conditional measures $\Pi(x) = \nu^x$, that is, if $\int f(y) d\nu^x(y) = 0$ for μ -almost all x implies $f \equiv 0$. Below we will give an equivalent definition.

We will not continue the list of definitions and restrict ourselves only with notions we need in this paper. For example, we do not consider the entropy of polymorphisms, spectral properties, etc.

2.3. Why polymorphisms? The notion of polymorphism widely extends the theory of transformations with invariant measure. We briefly illustrate some advantages and aspects of this notion.

1. A polymorphism of a finite measure space X with the uniform measure $\mu(\cdot) = 1/\#X$ to itself is a bistochastic matrix of order $m = \#X$. Such matrices form the semigroup $V_m = \{(a_{i,j})_{i,j=1}^m : \sum_j a_{i,j} = \sum_i a_{i,j} = 1/m, a_{i,j} \geq 0\}$.⁵ Thus the factor polymorphism of any polymorphism by a finite partition with n parts can be identified with a bistochastic matrix of order n .

This leads to the following easy proposition.

Proposition 2. *Every polymorphism of a continuous measure space can be represented as the inverse limit of a sequence of polymorphisms of finite spaces with uniform measures, or, in other words, of bistochastic matrices, for example, of orders 2^n . The semigroup \mathcal{P} of all polymorphisms of a continuous measure space with the weak topology is the inverse limit of a sequence of semigroups of bistochastic matrices: $\mathcal{P} = \text{projlim}_n V_{2^n}$.*

This gives an alternative approach to approximation in the ergodic theory of automorphisms. For some reasons, approximation of automorphisms with positive entropy by bistochastic matrices (= “periodic” polymorphisms) is more natural than approximation by periodic automorphisms. Following the physical terminology, we can call this type of approximation the *coarse-graining approximation of dynamical systems* (see [16]). I think that it is a fruitful method of studying K -automorphisms.

2. Polymorphisms allow us to extend ordinary notions in a natural way. For example, the conjugate to an endomorphism does not exist in the ordinary sense, but does exist as a polymorphism; we called such a polymorphism an *exomorphism*: in this case, a point has several images but one preimage.

As we have seen, the notion of polymorphism allows us to consider quotients of arbitrary automorphisms by arbitrary measurable partitions.

We may say that the theory of polymorphisms is the envelope of the theory of endomorphisms with respect to extending the notion of factorization from invariant partitions to arbitrary ones.

Moreover, any two partitions, say ζ and η , produce a polymorphism with invariant measure:

$$(X_\zeta, \mu_\zeta) \longleftarrow (X_{\zeta \vee \eta}, \nu_{\zeta \vee \eta}) \longrightarrow (X_\eta, \mu_\eta),$$

where all three spaces are the quotients of the space (X, μ) by the corresponding partitions. We will use this remark later.

⁵It is convenient to assume that the rows of matrices sum to $1/m$ and not to 1 as usual.

In the special case when we have one partition ξ (and the second one is the partition into separate points) we have the “tautological” polymorphism from X/ξ to X , which associates with an element $C \in X/\xi$ the conditional measure on $C \in X$.

3. Polymorphisms and correspondences. The simplest classical source of polymorphisms with finitely many images and preimages is a *correspondence* in the sense of algebraic geometry; the following scheme gives the simplest example. Consider the 2-torus $\{\mathbb{T}^2 = (u, v) : u, v \in \mathbb{C}, |u| = |v| = 1\}$ and the curve

$$u^n = v^m, \quad n, m > 1,$$

equipped with the Lebesgue measure. The conditional measures (transition and cotransition) are the uniform measures. We obtain a polymorphism of the unit circle with the Lebesgue measure to itself. The dynamics of such polymorphisms is very interesting and still poorly studied. This example can also be regarded as a bifibration over the circle.

4. Polymorphisms and Markov processes. Below we will describe the link to the theory of Markov processes: a polymorphism is the two-dimensional distribution of a stationary Markov process. The theory of polymorphisms leads to a new kind of examples and problems (see the second part of the paper) and help to understand the structure of general Markov processes.

5. Random walks on automorphisms as polymorphisms. Another typical example of polymorphisms came from the theory of random walks: in this case, a polymorphism is a (finite or infinite) convex combination of deterministic transformations, for example, shifts on some group of measure-preserving transformations; the coefficients of this convex combination may depend on points: assume that $\{T_\alpha, \alpha \in A\}$ is a family of transformations with quasi-invariant measure μ ; then $\Pi(x) = \mu^x$, where μ^x is a measure on the set $\{T_\alpha x : \alpha \in A\}$, or, better to say, μ^x is a measure on the set of parameters A that depends on x . For Π to be a polymorphism with invariant measure, these measures must satisfy some conditions.

6. The theory of **polymorphisms** is closely related to the theory of **joinings**, which are nothing more than polymorphisms with additional symmetries (for example, commuting with the automorphism $T \times T$ of the space $(X \times X, \nu)$ in the above notation). We can also say that a polymorphism with invariant measure is a joining of identical maps. It is more important that the quasi-similarity of two automorphisms can be also formulated as a problem on joinings of special type.

7. Orbit partition of a polymorphism. The *trajectory partition*, or *orbit partition* of a polymorphism Π of a space (X, μ) is defined as follows: two points x, y belong to same orbit if and only if there exist positive integers n, m such that the measures $\Pi^n(x)$ and $\Pi^m(y)$ are not mutually singular as measures on X . Denote by $o(x)$ the orbit of a point x under the polymorphism Π . If the polymorphism is of discrete rank (see the definition above), then the orbit partition has countable fibers. In this case, we obtain a new wide class of nonmeasurable partitions, or ergodic equivalence relations; a very intriguing question is to find a criterion of hyperfiniteness (tameness) of these partitions or to study their properties in terms of polymorphisms.

8. Polymorphisms and groupoids. This is a very important link. For simplicity, assume that an ergodic polymorphism Π is of discrete rank. Then its orbit partition defines an ergodic equivalence relation and a measurable groupoid (see [12]). We call Π *complete* if the measure $\Pi(x)$ is strictly positive on the orbit

$o(x)$ for almost all x ; in this case, the Π -image of x is a measure on the whole orbit of x , or, in other words, the bistochastic measure of Π is positive on the groupoid.

2.4. Quasi-similarity of polymorphisms and automorphisms. A classification of polymorphisms (or Markov operators, see below) can be defined in many ways. One of them is the classification up to conjugation with respect to a measure-preserving automorphism (see [16] for discussion). In this paper we will consider the classification up to quasi-similarity.

Definition 3. A polymorphism (in particular, auto- or endomorphism) Π_1 is a *quasi-image* of a polymorphism Π_2 if there exists a dense polymorphism Γ such that

$$\Gamma \cdot \Pi_1 = \Pi_2 \cdot \Gamma.$$

We say that two polymorphisms are *quasi-similar* if each of them is a quasi-image of the other.⁶

Question. To describe the notion of quasi-similarity for measure-preserving auto- and endomorphisms: does it coincide with the notion of isomorphism? It is especially important to know the answer for K -automorphisms.

But we will study the special case when T is a measure-preserving automorphism of a Lebesgue space (X, μ) and Π is a polymorphism with invariant measure of the same space.

Problem: *When does exist a dense polymorphism Γ of (X, μ) such that*

$$\Gamma \cdot \Pi = T \cdot \Gamma \quad ?$$

A similar problem: *when does exist a dense polymorphism Λ such that*

$$\Pi \cdot \Lambda = \Lambda \cdot T \quad ?$$

Or when the automorphism T and the polymorphism Π are quasi-similar?

Looking ahead and using the notions that will be introduced later, we can say that, in order to avoid trivial cases (when both T and Π are automorphisms), we should suppose that Π is *prime* (= has no nontrivial factor endomorphisms, or has no nontrivial invariant partitions). Furthermore, a mixing polymorphism cannot be quasi-similar to any measure-preserving transformation of a continuous measure, thus we may assume without loss of generality that Π (or Π^*) is nonmixing; this means that $\Pi^n \rightharpoonup \Theta$ (respectively, $\Pi^{*n} \rightharpoonup \Theta$) in the weak topology as $n \rightarrow \infty$, where Θ is the zero polymorphism.

3. The operator formalism and Markov operators. In this section we consider two alternative languages for the theory of polymorphisms: the first one is the *operator formalism* in the space of measurable square integrable functions $L^2_\mu(X)$, the language of so-called *Markov operators*, and the second one is the language of *stationary Markov processes*, which is especially important for polymorphisms.

⁶The density of Γ in this definition is a very important condition; without it, the equivalence is trivial.

3.1. Markov operators. The functional analog of the notion of polymorphism is the notion of *Markov operator* in some functional space, which in this paper will be the Hilbert space $L^2_\mu(X)$.

Definition 4. A linear operator V in $L^2_\mu(X)$ is called a *Markov operator* if

- 1) V is a contraction: $\|V\| \leq 1$ in the operator norm;
- 2) $V1 = V^*1 = 1$;⁷
- 3) V is positive, which means that Vf is a nonnegative function provided that $f \in L^2_\mu(X)$ is nonnegative.

It is easy to prove that the set \mathcal{M} of all Markov operators is a convex weakly compact semigroup with involution $V \rightarrow V^*$.

Unitary (isometric) Markov operators are precisely the operators generated by measure-preserving auto(endo)morphisms. We generalize this correspondence to polymorphisms.

Proposition 3. 1. *Let Π be a polymorphism of a space (X, μ) with invariant measure; then the formula*

$$(W_\Pi f)(x) = \int_X f(y) \mu^x(dy)$$

defines correctly a Markov operator in L^2 .

2. *Every Markov operator W in the space $L^2_\mu(X)$, where (X, μ) is a Lebesgue space with continuous finite measure, can be represented in the form $W = W_\Pi$, where Π is a polymorphism of (X, μ) with invariant measure.*

3. *The correspondence $\Pi \mapsto W_\Pi$ is an anti-isomorphism between the semigroup with involution of mod0 classes of polymorphisms and the semigroup of Markov operators; this correspondence is also an isomorphism of convex compact spaces.*

The proof follows from the standard theorems of functional analysis (see [6]), and we mention only the formula for the inverse map from the semigroup of Markov operators to the semigroup of polymorphisms (for more details, see [16]). Let U be a Markov operator; the bistochastic measure of the corresponding polymorphism is defined as follows:

$$\mu(B_1 \times B_2) = \langle U\chi_{B_1}, \chi_{B_2} \rangle.$$

The check of all required assertions is automatic.

Note that the correspondence $\Pi \mapsto W_\Pi$ is a contravariant correspondence and reverses the arrows.

We will denote by W_Π, W_Λ, \dots the Markov operators corresponding to polymorphisms Π, Λ, \dots . If Π is an automorphism, then the operator W_Π is unitary. It is clear that this correspondence extends the ordinary correspondence $T \mapsto U_T$, $(U_T f)(x) = f(T^{-1}x)$, between measure-preserving automorphisms and unitary multiplicative real operators (= automorphisms of the unitary ring).

The compact space \mathcal{M} is the convex weakly closed hull of the group of unitary multiplicative real operators.

The orthogonal projector $\mathbf{1}$ onto the one-dimensional subspace of constants is the Markov operator corresponding to the zero polymorphism: $\mathbf{1} = W_\emptyset$.

An equivalent and more useful definition of the Markov operator corresponding to a polymorphism is as follows. Consider a bistochastic measure ν on the space

⁷This condition expresses the invariance of the measure under the polymorphism (see below); the equality $V^*1 = 1$ follows automatically from the other conditions.

$X \times X$ and the Hilbert space $L^2_\nu(X \times X)$. Consider two subspaces H_1 and H_2 in this space that are the images of $L^2_\mu(X)$ under the embedding of the spaces of functions of the first and second arguments, respectively, to the whole space $L^2_\nu(X \times X)$:

$$H_1 = L^2_\mu(X) \longrightarrow L^2_\nu(X \times X) \longleftarrow L^2_\mu(X) = H_2.$$

Denote the orthogonal projection onto the subspace H_i by P_i , $i = 1, 2$; then the above definition of Markov operators coincides with the following one.

Proposition 4.

$$\begin{aligned} W_\Pi f &= P_2 \cdot P_1 f, & f \in H_2; \\ W_{\Pi^*} g &\equiv (W_\Pi)^* g = P_1 \cdot P_2 g, & g \in H_1. \end{aligned}$$

It is worth mentioning that the conditional expectation, or orthogonal projection, onto the subalgebra of functions that are constant on elements of a measurable partition is the Markov operator corresponding to the “tautological” polymorphism, which we defined in the previous section.

3.2. Properties of polymorphisms in terms of Markov operators. It is not difficult to reformulate all the notions introduced for polymorphisms (ergodicity, mixing, primality, density, etc.) in terms of Markov operators.

First of all, the mean and pointwise ergodic theorems for Markov operators have the following form (this is an old result by Hopf and Chacon–Ornstein, see [11]):

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (W_\Pi^k f)(x) = P f,$$

where P is the projection onto the maximal fixed subspace (subalgebra); $P = \Theta$ if the polymorphism Π is ergodic.

A Markov operator W corresponds to a *mixing polymorphism* Π if and only if the sequence W^n weakly tends, as $n \rightarrow \infty$, to the projection onto the subspace of constants:

$$W^n \rightarrow \mathbf{1} = W_\Theta.$$

We will discuss this property in detail later.

We will call a Markov operator $W = W_\Pi$ *dense* if the W -image of $L^2_\mu(X)$ is dense in $L^2_\mu(X)$. Obviously, $W = W_\Pi$ is dense if and only if the corresponding polymorphism W is dense. The density of W is equivalent to the following condition: the conjugate operator W^* has zero kernel. In the terminology of the book [10], a contraction with dense image is called a quasi-affinitet.

Definition 5. A Markov operator V is called *totally nonisometric* if there is no nonzero subspace in the orthogonal complement to the subspace of constants in $L^2_\mu(X)$ on which W is isometric.

Proposition 5. *A Markov operator is totally nonisometric if and only if it corresponds to a prime polymorphism.*

We are interested in Markov operators that are far from isometries (in other words, in polymorphisms that are far from automorphisms). Of course, a mixing Markov operator is totally nonisometric, but at the same time it is not true that every Markov operator is the direct sum of a mixing and isometric Markov operators. Our main examples will illustrate this effect.

Recall the following notation from operator theory (see [10]), which we use for the case of Markov operators.

Definition 6. A contraction W acting in a Hilbert space H belongs to the classes $C_{0,\cdot}$, $C_{\cdot,0}$, $C_{1,\cdot}$, $C_{\cdot,1}$ if for every function $f \in H$ that is orthogonal to the subspace of constants we have $W^n f \rightarrow 0$, $W^{*n} f \rightarrow 0$, $W^n f \not\rightarrow 0$, $W^{*n} f \not\rightarrow 0$, respectively. The classes $C_{a,b}$, $a, b = 0, 1$, are defined in an obvious way. All these classes are nonempty.

We are interested mainly in Markov operators in $L^2_\mu(X)$ of the class $C_{1,\cdot}$ (or $C_{\cdot,1}$, or $C_{1,1}$), which are totally nonisometric and, consequently, correspond to nonmixing prime polymorphisms (respectively, polymorphisms whose conjugates are prime nonmixing; or polymorphisms such that both the polymorphism and its conjugate are prime nonmixing). This class is also the most interesting from the viewpoint of the pure operator theory of contractions. In [10] it was proved that contractions of type $C_{1,1}$ are quasi-similar to unitary operators; we will extend this fact to Markov operators in Section 4.

The existence of totally nonisometric nonmixing Markov operators is not *a priori* obvious. We will describe all such examples. The main feature of such examples is that they are not the direct products of mixing and pure deterministic operators.

The convex structure of the set of all polymorphisms \mathcal{P} and the isomorphic compact set of Markov operators \mathcal{M} is very important. Isometries and unitary operators are extreme points of this compact set, but there are many other extreme points. In [16] it was proved that the set of extreme polymorphisms is an everywhere dense G_δ -set in \mathcal{P} . These extreme Markov operators (polymorphisms) have many interesting properties (see [16], [15]).

From the viewpoint of the theory of C^* -algebras, it is natural to consider the C^* -algebras generated by some class of multipliers (say, continuous functions) and a given Markov operator and its conjugate; this is a generalization of the ordinary notion of cross product (with an action of the group \mathbb{Z}) and cross products with endomorphisms (see a recent paper [2]). One of the open questions concerns the amenability of the corresponding C^* -algebra.

Let us also mention the following problem, which was formulated in [18]:

Problem. To characterize the C^* -algebra $\text{Alg}(\mathcal{M})$ generated by all Markov operators in $L^2_\mu(X)$. This (nonseparable) algebra does not coincide with the algebra $B(L^2_\mu(X))$ of all bounded operators.⁸ On the other hand, this algebra is distinguished and plays the same role in measure theory and the theory of Markov operators as the algebra of bounded operators $B(H)$ plays in operator theory.

3.3. The operator formulation of quasi-similarity.

Definition 7. A Markov operator W_1 is called a *quasi-image* of a Markov operator W_2 if there exists a dense Markov operator U such that $U \cdot W_1 = W_2 \cdot U$. Markov operators W_1 and W_2 are called *quasi-similar* if each of them is a quasi-image of the other.

As follows from definitions, two polymorphisms Π_1, Π_2 are quasi-similar if and only if the corresponding Markov operators are quasi-similar. The same is true for quasi-images.

⁸For example, in $L^2(\mathbb{R}, m)$ (where m is the infinite Lebesgue measure), the operator of Fourier transform does not belong to this algebra, as was observed by G. Lozanovsky, see [18].

Now let us define more accurately the problem of *quasi-similarity of automorphisms and polymorphisms*, which was formulated above in terms of operator formalism. Denote by U_T the unitary operator in $L^2_\mu(X)$ corresponding to an automorphism T and by W_Π the Markov operator in $L^2_\mu(X)$ corresponding to a polymorphism Π ; as we have seen, Π is prime if and only if W_Π is *totally nonisometric*, i.e., W_Π has no invariant subspaces (except the one-dimensional subspace of constants) on which it acts as an isometry. We also may assume without loss of generality that $W_{\Pi^n} \rightharpoonup P$ (in the weak topology; here P is the projection to the subspace of constants), because in our case Π is nonmixing. Under this condition, our problem is formulated as follows:

Problem. When the Markov operator U_T can be a quasi-image of the Markov operator W_Π and vice versa? When they are quasi-similar in the sense of the previous definition?

4. Markov processes associated with polymorphisms and dilations of Markov operators.

4.1. Markov processes. Let Π be a polymorphism of a space (X, μ) with invariant measure, and let ν be the corresponding bistochastic measure on $X \times X$. As we have mentioned above, every polymorphism with invariant measure generates a stationary Markov process; thus we consider ν as the *two-dimensional distribution* of a stationary Markov process $\Xi(\Pi)$. For this process, X is the state space and μ is an invariant one-dimensional distribution. Denote by M the Markov measure in the space $\mathcal{Y} \equiv X^{\mathbb{Z}}$ of realizations of the process $\Xi(\Pi) \equiv \{\xi_n, n \in \mathbb{Z}\}$, and by $S = S_\Pi$ the right shift in the measure space (\mathcal{Y}, M) . The space $L^2_\mu(X)$ is naturally embedded into the space $L^2_M(\mathcal{Y})$ as the subspace of functionals of realizations of the process that depend only on the value ξ_0 of the process at time zero. The state space at time n , which is identical to X , will be denoted by X_n . Let S be the right shift in the space (\mathcal{Y}, M) ; it preserves the measure M and is called the *Markov shift* corresponding to the polymorphism Π . For example, if $\Pi = \Theta$, then the Markov shift is a Bernoulli shift.

Note that the Markov process corresponding to the conjugate polymorphism Π^* is obtained from the Markov process of Π by reversing time.

Recall (see [10]) that if W is a contraction acting in a subspace $L \subset H$ of a Hilbert space H , then a unitary operator U acting in H is called a *dilation* of W if

$$W^n = PU^n,$$

where P is the orthogonal projection $P : L \rightarrow H$. Every contraction has the so-called minimal dilation (see [10]).

Definition 8. A *dilation* of a Markov operator W in $L^2_\mu(X)$ is a Markov operator U in some space $L^2_\alpha(Y) \supset L^2_\mu(X)$ such that

$$W^n = PU^n,$$

where P is the (positive) orthogonal projection $P : L \rightarrow H$.

Proposition 6. *The unitary operator U_S in $L^2_M(\mathcal{Y})$ is a Markov dilation of the Markov operator W_Π , which acts in the space $L^2_\mu(X_0)$ regarded as a subspace of $L^2_M(\mathcal{Y})$:*

$$W_\Pi = P_0 U_S.$$

Here P_0 is the expectation (orthogonal projection) onto $L^2_\mu(X_0)$. This dilation is not the minimal dilation of W_Π in the sense of operator theory, but it is the minimal **Markov** dilation (see [16]).

A general problem is to characterize invariant properties of the Markov shift S in terms of the polymorphism (= Markov generators), for example, to give a characterization of the Bernoulli and non-Bernoulli properties of Markov shifts, or to describe relations between regular Markov processes and K -automorphisms, etc.

4.2. Mixing, primality, and tail σ -field of Markov processes. It is clear that the ergodicity of a polymorphism Π is equivalent to the ergodicity of the process $\Xi(\Pi)$ and to the ergodicity of the Markov shift S , which is an invariant property.

Contrary to this, mixing and primality and other properties of a polymorphism and the corresponding Markov process are not invariant properties of the Markov shift regarded as an abstract measure-preserving transformation, but can vary for different generators.

Assume that Π is a polymorphism of (X, μ) and $\Xi = \Xi(\Pi) = \{\xi_n\}_{n \in \mathbb{Z}}$ is the corresponding Markov process with state space X and Markov measure M in $\mathcal{Y} = X^{\mathbb{Z}}$. Denote by \mathfrak{A}_n the σ -subfield in \mathcal{Y} generated by the set of one-dimensional cylindric sets at time n , and by \mathfrak{A}_- (respectively, \mathfrak{A}_+) the tail σ -field of the past (respectively, future) of the process.⁹ Denote the corresponding partitions into infinite pasts (futures) by $\xi_{\mp\infty}$, and the quotient spaces with measures (the “infinite past” and the “infinite future”) by $(X_{\mp}, M_{\mp}) = (X^{\mathbb{Z}}, M)/\xi_{\mp\infty}$. These spaces can also be called the *infinite entrance boundary* and *exit boundary*.

Let $S = S_\Pi$ and S^{-1} be the right and left shifts, respectively, in the space (\mathcal{Y}, M) .

Recall that a stationary process (even not necessarily Markov) is called *regular* or *pure nondeterministic* in the past (future) if the tail σ -field of the past (future) (or the entrance (exit) boundary) is trivial.¹⁰ Let us emphasize that this is not an invariant property of the shift, but a property of the generator (process).

If the tail σ -field is not trivial, then almost every point $x_{\mp} \in X_{\mp}$ determines the conditional Markov (nonhomogeneous in time) process $\{\xi_n^{x_{\mp}}\}$. The correspondence $x_{\mp} \mapsto \{\xi_n^{x_{\mp}}\}$ determines the decomposition of the whole space (\mathcal{Y}, M) and the process $\{\xi_n\}$ into a direct integral over the quotient space X_{\mp} . We can correctly define the conditional measure on the σ -field \mathfrak{A}_0 as the one-dimensional distribution at moment 0 of the conditional Markov process.

Now we can summarize the information on the tail σ -fields of Markov processes and mixing in the following theorem. We formulate it only for the σ -field of the past; the same is true for the future.

Theorem 1. *For a Markov process $\Xi = \Xi(\Pi)$, the following assertions are equivalent:*

- 1) *The process Ξ is regular in the past, which means that the σ -field \mathfrak{A}_- is trivial, i.e., X_- is a one-point space.*
- 2) *The limit (which exists with probability 1) of the conditional measures*

$$\lim_{n \rightarrow -\infty} \Pr\{a | x_n\}, \quad a \in \mathfrak{A}_0, \quad x_n \in X_n,$$

on \mathfrak{A}_0 does not depend mod 0 on the trajectory $\{x_n\}_{n \in \mathbb{Z}}$ and coincides with the unconditional measure.

⁹Or the intersection over all positive n of the σ -fields generated by the values of the process before time $-n$ (respectively, after time n).

¹⁰The term “regularity” in this sense was first used by Kolmogorov.

- 3) *The Markov generator ξ_0 is a K -generator for the right shift S .*
 4) *The polymorphism Π is mixing, i.e., $\Pi^n \rightarrow \Theta$.*

The equivalence of the first three claims follows more or less from definitions and, in contrast to the equivalence with claim 4), does not use the Markov property. The equivalence between 3) and 4) for Markov processes is well known and can be proved directly.

For Markov chains, i.e., processes with *finite or countable state space*, and for some special cases of polymorphisms (= transition probabilities), the mixing property is equivalent to the property that is usually called the “absence of nontrivial subclasses of the state space” (or irreducibility, or convergence of the powers of the transition matrix to an invariant vector, etc.; see, e.g., [1]) and that in this paper we have called “primality” — the absence of nontrivial factor endomorphisms — or, equivalently, to the following property: a Markov process has no nontrivial deterministic quotients.

Thus for Markov chains with discrete state space, we can add to the above theorem the following fifth claim, which is equivalent to 1)–4) in this case, but is not equivalent to them in the general case:

- 5) *There are no nontrivial partitions of the state space invariant with respect to the matrix of transition probabilities* (see [20]).

Definition 9. A Markov process corresponding to a prime nonmixing polymorphism will be called *quasi-deterministic*.

M. Rosenblatt was perhaps the first to point out the existence of quasi-deterministic (“paradoxical”) Markov processes (see [14, 4.4]). A more general construction for Anosov systems was suggested in [17] and was called *superstability*.

Thus such a process is not regular, but has no deterministic factors. Note that in the case of a quasi-deterministic Markov process any measurable set from the σ -fields \mathfrak{A}_{\mp} of measure not equal to 0 or 1 is not a cylindric set (see also [17]). This also contrasts with the theory of Markov chains, where such sets are one-dimensional cylinders. In the last section we give a description of quasi-deterministic Markov processes; it turned out that precisely these polymorphisms are quasi-similar to K -automorphisms.

5. Quasi-similarity of automorphisms and polymorphisms and the K -property.

5.1. The structure of a quasi-deterministic Markov process. Here we briefly describe the structure of the past of a quasi-deterministic Markov process.

It is well known (see, e.g., [13]) that for any stationary process with discrete time (even not necessarily Markov) there is a *canonical automorphism that acts on the tail σ -fields \mathfrak{A}_{\pm} and on the quotient spaces X_{\pm}* : this is the restriction of the left shift S^{-1} (respectively, right shift S_+) to this σ -field and, consequently, to the quotient spaces X_{\pm} ; it is called the *residual (tail) automorphism*.¹¹

Denote these automorphisms by S_{\mp} (it is convenient to use the left shift S^{-1} in the past and the right shift S in the future). Now we can formulate the first theorem on interrelations between the past and present.

¹¹In ergodic theory, it is sometimes called the Pinsker automorphism, and the σ -field \mathfrak{A}_{\pm} is called the Pinsker σ -field.

Theorem 2. Assume that Π is a prime nonmixing polymorphism. Let $\Xi = \{\xi_n\}_{n \in \mathbb{Z}}$ be the quasi-deterministic stationary Markov process associated with Π . Then

1. The tail σ -field \mathfrak{A}_- of the process is not trivial. The tail (residual) automorphism S_- acting on the space (X_-, M_-) is ergodic.
2. Define a polymorphism Λ' as follows: the value $\Lambda(x_-)$, $x \in X_-$, is the conditional measure μ_{x_-} on the state space X_0 corresponding to the point x_- of the tail space X_- . Then

$$\Pi \cdot \Lambda' = \Lambda' \cdot S_-, \tag{1}$$

in other words, S_- is quasi-similar to the polymorphism Π .

The polymorphism Λ' is injective (see the definition in Section 2).

3. The conjugate polymorphism Λ'^* from the space (X_0, μ) to the tail space (X_-, M_-) is also injective; its value at a point $x \in X_0$ is the conditional measure on the infinite past X_- given that the value of the process at zero time is equal to x .

4. There exists an isomorphism Q between the state space (X_0, μ) and the infinite past (tail space) (X_-, M_-) , which determines an automorphism T of the state space (X_0, μ) by the formula $T = Q^{-1}S_-Q$ and a polymorphism of this space by the formula $\Lambda = \Lambda'Q$ such that the automorphism T is a quasi-image of the polymorphism Π :

$$\Lambda \cdot T = \Pi \cdot \Lambda.$$

In our terminology, the last formula means that T is a quasi-image of Π ; if the polymorphism Π^* is also prime and nonmixing, then we have an analogous formulation with the tail σ -field of the “future” and obtain the *quasi-similarity* between T and Π . We omit the proof and observe that the main part of the theorem is the “Markov” or “ergodic” analog of the corresponding theorem on contractions in Hilbert spaces (see [10, Ch. 2]) with some serious complications. Indeed, we consider two subspaces in $L^2(\Xi, M)$: the space of functions measurable with respect to the tail σ -field and the state space at zero moment; these subspaces generate the polymorphism Λ' (see Section 2).

Corollary 1. We can express the polymorphism Λ directly in terms of the main ingredients T and Π :

$$\Lambda = \lim_{n \rightarrow \infty} \Pi^n T^{-n}. \tag{2}$$

The last theorem reduces the quasi-similarity between an automorphism and a polymorphism to the state space (X, μ) ; the role of the tools of the theory of Markov processes is simply in using the residual automorphism and interlacing polymorphisms (conditional measures). Below we will prove that T is a K -automorphism.

For our purposes, it is very convenient to represent the polymorphism Π as the product

$$\Pi = \Phi \cdot T,$$

where Φ is a new polymorphism; then it is easy to check that

$$\Lambda = \lim_{n \rightarrow \infty} \Phi \cdot T \Phi T^{-1} \dots T^n \Phi T^{-n};$$

If we set $\Phi_k = T^k \Phi T^{-k}$, $k \in \mathbb{Z}$, $\Phi \equiv \Phi_0$, then

$$\Lambda = \lim_{n \rightarrow \infty} \prod_0^n \Phi_k = \prod_0^\infty \Phi_k.$$

In this setting, the main formula $\Pi \cdot \Lambda = \Lambda \cdot T$ takes the following form:

$$\Phi_0 \cdot T \cdot \prod_0^n \Phi_k = \prod_0^n \Phi_k \cdot T.$$

The convergence of the infinite product is the only condition for the construction to be well-defined; in the above situation, this follows from the existence of the conditional measures on X_0 with respect to the infinite past. In the examples of the next section it will be proved directly.

As a result of this section, *for every nonmixing prime polymorphism Π of a space (X, μ) we have found (using the corresponding Markov process) an automorphism T such that*

$$\lim_{n \rightarrow \infty} \Pi^n T^{-n} = \Lambda$$

and equation (1) holds. The Markov property of the processes was used to reduce the problems to a single space (the state space).

Remark. The orbit partition of the polymorphism Λ is an analog of the partition into subclasses in the theory of Markov chains. From the above formulas we can conclude that *it is an ergodic (absolutely nonmeasurable) equivalence relation* (the same is true for the polymorphism Λ^*). Indeed, on the one hand, its measurable hull is an invariant partition for Π , hence it is the trivial partition; but on the other hand, the orbit partition is not equal to the trivial partition, because the polymorphism Π is nonmixing; consequently, this is an absolutely nonmeasurable partition. We will use this property in the next section. This fact is crucial; it illustrates our previous remark on the role of the nonmeasurability of the partition into subclasses for general Markov processes.

The Markov shift we have considered obviously has a positive entropy and under natural assumptions is a K -automorphism (but the Markov generator is not a K -generator!). We will discuss this elsewhere.

5.2. Random perturbations of K -automorphisms. Using the symbolic representation of K -automorphisms (see, e.g., [7]), we can give a generalization of the construction from [17], which associates with any K -automorphism T a polymorphism that is a quasi-image of T , and also find polymorphisms for which T is a quasi-image. For the special case of Bernoulli automorphisms, it is possible to find polymorphisms that have both properties simultaneously and, consequently, are quasi-similar to T .

Theorem 3. *For every K -automorphism T there exists a prime nonmixing polymorphism Π_- such that T is a quasi-image of Π_- , and a polymorphism Π_+ that is a quasi-image of T . If there exists a symbolic realization of T with finite or countable state space in which the homoclinic equivalence relation is ergodic,¹² then there exists a polymorphism that is quasi-similar to T ; this is the case of Bernoulli automorphisms.*

The construction is more or less direct; the proof includes some combinatorial construction. For a given K -automorphism T , the set of such polymorphisms is very large, but it is possible to describe all such polymorphisms in terms of the symbolic

¹²Recall that two sequences $\{x_i\}$ and $\{y_i\}$ of the space of sequences with a shift-invariant measure belong to the same homoclinic class if for sufficiently large N we have $x_i = y_i$ for $i > N$ (see [5, 17]). It is not known to the author whether the homoclinic equivalence relation is ergodic for an arbitrary K -automorphism.

realization of T — these are nothing more than random perturbations of the K -automorphism along “stable,” “unstable,” or homoclinic (which is the intersection of a stable and an unstable one, if it is nonempty) leaves of the automorphism, respectively; each case gives the corresponding quasi-images or quasi-similarity. In [17], the following terminology was used: a K -automorphism is “super-stable” in the past (respectively, in the future, or both); the reason for the term is that after these random perturbations, the initial automorphism can nevertheless be recovered up to isomorphism as the tail automorphism in the past or future. In short, this means that *a random perturbation of a K -automorphism allows us to recover automatically the original automorphism.*

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